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A New Probabilistic "Paradox" Involving the Bayes Theorem

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Abstract

In this paper we present a new probabilistic "paradox" involving the Bayes theorem. The resolution of the paradox is elementary, but it has some subtly features due to the fine tuning of the integer values used in order to develop the working mechanism of the story. The author anticipates the "paradox" itself worth some attention because it seams to shed a light on several important issues, including the way sometimes precise information can be gained from highly uncertain data.

Keywords: probabilistic paradox, Bayes Theorem

Probabilistic paradoxes continue to challenge the today's mathematical literature. For example a paradox so simple as the Two-envelope problem has a very large literature, see for example the survey article [8], which includes treatment of some philosophical aspects, too. The paradox is the following. A player must make a choice between two envelopes, one containing twice as much money as the other. After seeing the content of the chosen envelope, the player is offered the opportunity to exchange it for the other envelope. Depending on the way we compute the expected gain, switching may seem advantageous or not advantageous.

The internet page [1] contains a list of well known paradoxes, including the Monty Hall Problem, too. This problem – though very simple – is often cited in connection with the catch which can arise as counter-intuitive consequence of the conditional probability.

On the road toward the conditional probabilities the first paradox is the division paradox, first published in Venice in 1494 by Fra Luca Paccioli (see [9]). The correct solution of this problem was given only in 1654 by Pascal and Fermat, independently of each other. The problem says that two players are playing a fair game for a prize money, and they agreed that whoever wins 6 round first, gets the whole prize. Suppose the game is interrupted when the first player has won 5 and the second player has won 3 rounds. How should the prize be divided fairly among the two players? Is correct to divide the prize in the rate of the round wins, 5:3? Tartaglia suggested a division in the rate 2:1. His argument was that the first player won 2 more rounds then the second one, which is 1/3 of the whole number of 6 rounds, so he deserve 1/3 of the prize. The rest of 2/3 of the prize obviously has to be divided equally, hence the ratio of 2:1 of the prize division. However the correct answer and the fair ratio is 7:1 (see also [3] and [7]).

Paradoxes arrive already in connection with the product rule of probabilities, i.e. in connection with the notion of independence of events. Two events A and B are independent exactly when the probability of the event $A \cap B$ (i.e. both events occur) is the product of the probabilities of events A and B, $p(a \cap B) = p(A) \cdot p(B)$. The paradox of independence says that if we toss two coins – a red and a blue one – and denote A the event *the red coin falls head*, B the event *the blue coin falls head*, and C the event *one, and only one coin falls head*, then A, B, and C are pairwise independent but any two of them determine the third one. The resolution of this paradox is that pairwise independence do not mean global independence, i.e. the product rule do not extends for three or more events under the assumption of only the pairwise independence of the events (see also [4],[6] and [10]).

There exists also a whole book on the probabilistic paradoxes, [9]. The book contains problems that can be considered as probabilistic paradoxes in a broder sens, such as problems with highly counterintuitive solutions. Our problem – in the author's opinion – deserve the standard of such a "paradox". It is on the way along the paradoxes if independence and the conditional probabilities, and it speculates some subtleties related to the Bayes theorem.

Let us present the paradox itself (it was inspired by [2]).

1 The paradox

A taxicab is involved in an accident, in a city at nighttime . The only witness declares that the taxicab was red-colored. In the city there are 85% red and 15% green taxicabs. The insurance company claims a sight-test for the witness, and therefore he is tested by showing him photos of red and green colored taxicabs for short time, checking his ability to recognize the proper color. Surprisingly, it is found out that he correctly recognizes only 80% of the colors, as in 20% of test cases he mixes up colors, saying the cab is red when it is actually green or vice-versa. As a detail, it is established that the green color is viewed as red in no more then 20% of the test cases.

On the other hand the lawyer of the insurance company claims at least 95% certainty, consequently the company refuses to pay the damage.

The aggrieved party brings a suit against the company, so the judge orders a precise analysis of these data. An independent expert (a mathematician of course) makes the appropriate computations and concludes that if we assume the witness's statement, then the cab was red colored with a probability of at least 95%!

Question A. *How did the expert come to this conclusion?*

It is understandable that the advocate of the insurance company did not accept the expert's report. He had multiple arguments. First he discovered in the meantime that the data used in the computation are not quite accurate: as a matter of fact in the city there are only 84% of red, 13% of green and in addition to these a 3% of blue colored taxicabs, too. More importantly, he claimed that the witness is probably colorblind, as he often mixes up the red and green colors. And finally, he pointed out that due to the high amount of the money which would be paid for the damage, the company's rules demand a much higher trustiness, more exactly at least 99% reliability of the red color, based on the witness's statement. Consequently, a more accurate color test is needed, which must check the blue color, too.

A new, more detailed color sight-test is performed, and the hunch of the advocate seems to be sustained by its outcome. The witness can be thought without doubt as being colorblind: the red color is correctly identified by him even less, in only 77% of test cases, and even worth, the green color is misidentified as red in 3% of cases and also the blue color is viewed as red in 8% of test cases!

The new test data are passed to the expert. He makes some new evaluations and declares two things. Firstly, the new test results accurately reconfirm the conclusion of the first test. Secondly, if we accept the testimony of the witness – which was an adjuration – then it can be declared that the cab was a red one, with more than 99% probability!

Question B. *Is the expert right again?*

The lawyer was disappointed and puzzled, but he said he had to accept the facts proved by the calculations. One last time he took a look at the test results, and stuck his eyes on the data, according to which the witness could hardly perceive the blue color, since he only identified it correctly in 10% of test cases! He remarked accordingly, that if the witnesses had sensed a blue taxi, then certainly it would have been easy to prove the wrong perception, and would have won the trial. On the contrary - the expert informed - if in the witness's affidavit had contained a statement about a blue taxi, then with the full 100% of confidence level, we would have known that the taxi that caused the accident was a blue one!

Question C. *How is this possible?*

2 Some preliminaries and notations

Let us recall quickly some computational rules of the probabilities and also the Bayes theorem.

Let A and B be two events, and let $A \cap B$ the event that both A and B take place. Then if we denote the probability of event A and B by $p(A)$ and $p(B)$ respectively, the probability of the event $A \cap B$ by $p(A \cap B)$, then

$$p(A|B) = \frac{p(A \cap B)}{p(B)}$$

is the so called *conditional probability* of A with respect to the event (hypothesis) B, which measures the probability of the event A assuming the event B occurred. If two out of the three probabilities in the above relation are known, or can be computed, then the third one can be computed, too. Also from this relation it follows the *product rule*

$$p(A \cap B) = p(B)p(A|B) = p(A)p(B|A).$$

Let us consider a *complete* set $(H_1, H_2, \ldots, H_n)$ of events, that is a set of events which are *mutually incompatible* (i.e. $H_i \cap H_j$ ="impossible", $i \neq j$) and *collectively exhaustive* ($H_1 \cup H_2 \cup \ldots \cup H_n$ ="certain"). They are intended to be viewed as some hypothesises that can have probabilistic influences on the event A. If these influences can a priori be evaluated through the conditional probabilities $p(A|H_i)$, then – and this is the essence of the Bayes Theorem – we can evaluate the a posteriori probabilistic "contribution" of the hypothesis H_i for the event A to occur, as

$$p(H_i|A) = \frac{p(H_i)p(A|H_i)}{p(A)},$$

where the denominator is the probability of A,

$$p(A) = p(H_1)p(A|H_1) + \ldots + p(H_n)p(A|H_n).$$

Now let us introduce some notations. We denote by

R	the car was Red colored
G	the car was Green colored
B	the car was Blue colored
R_p	the car was Perceived as being Red colored
G_p	the car was Perceived as Green colored
B_p	the car was Perceived as Blue colored
C	the witness perceived the colors Correctly
W	the witness perceived the colors Wrongly

The probabilities $p(R), p(G), p(B)$ are known through the distribution of colors among the taxicabs in the city, and the conditional probabilities of type $p(R_p|R), p(R_p|G), \ldots$ are given by the sight-tests of the witness.

It is clear that the task is to evaluate the conditional probability $p(R|R_p)$.

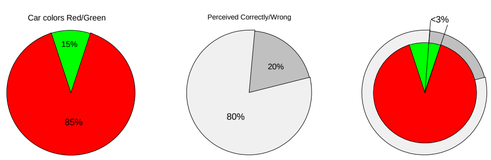
3 The paradox explained

The answer to the Question A.

In order to answer the first question, let us denote by a the unknown probability the witness perceive a red car as a red one, which is the conditional probability $p(R_p|R)$ and similarly by b the probability to perceive a green car as a red one, which is the conditional probability $p(R_p|G)$. Then we have the table

	R_p	G_p
R	a	$1-a$
G	b	$1-b$

Our aim is to compare the red-green car distribution with the correct-wrong color perception, which is shown in the next figure.



The car is perceived correctly either if it is red and perceived red or it is green and is perceived green.

$$C = (R \cap R_p) \cup (G \cap G_p). \tag{1}$$

Similarly the car is perceived wrong either if it is red but (i.e. and) is perceived green or it is green but (i.e. and) is perceived red.

$$W = (R \cap G_p) \cup (G \cap R_p). \tag{2}$$

Hence we have the next two equations:

$$\begin{aligned} p(C) &= p(R)p(R_p|R) + p(G)p(G_p|G) & (3)\\ p(W) &= p(G)p(R_p|G) + p(R)p(G_p|R). & (4)\end{aligned}$$

Substituting the values of probabilities, we have

$$\begin{aligned} 85 \cdot a + 15 \cdot (1-b) &= 80 & (5)\\ 15 \cdot (1-a) + 85 \cdot b &= 20 & (6)\end{aligned}$$

These are not independent equations, each one is a consequence of the other, so we have one free parameter. Now, the car was red assuming it was perceived red, with the probability:

$$\begin{aligned} p(R|R_p) &= \frac{p(R \cap R_p)}{p(R_p)} = \frac{p(R)p(R_p|R)}{p((R_p \cap R) \cup (R_p \cap G))} = \\ &= \frac{p(R)p(R_p|R)}{p(R_p \cap R) + p(R_p \cap G)} = \\ &= \frac{p(R)p(R_p|R)}{p(R)p(R_p|R) + p(G)p(R_p|G)} = \\ &= \frac{0.85a}{0.85a + 0.15b} = \\ &= \frac{0.80 - 0.15(1-b)}{0.80 - 0.15(1-b) + 0.15b} = \\ &= \frac{0.65 + 0.15b}{0.65 + 0.30b} \end{aligned} \tag{7}$$

This is a decreasing function of b on the interval $[0, 1]$, starting from 1 for $b = 0$. We know that $b \leq 0.20$ so the smallest value of this conditional probability is for $b = 0.20$, which is 0.957746, so we have

$$p(R|R_p) > 0.95.$$

The question A is answered.

The answer to the Question B.

If we denote by $p(R_p|R), p(R_p|G), p(R_p|B)$ the conditional probabilities that a car is perceived as a red one, while it is red, green and blue respectively, the new data show us that

$$\begin{aligned} &p(R_p) = \\ &= p((R_p \cap R) \cup (R_p \cap G) \cup (R_p \cap B)) = \\ &= p(R_p \cap R) + p(R_p \cap G) + p(R_p \cap B)) = \\ &= p(R)p(R_p|R) + p(G)p(R_p|G) + p(B)p(R_p|B)) = \\ &= 0.84 \cdot 0.77 + 0.13 \cdot 0.03 + 0.03 \cdot 0.08 = \\ &= 0.6531 \end{aligned} \tag{8}$$

which implies

$$p(R|R_p) = \frac{p(R \cap R_p)}{p(R_p)} = \\ = \frac{p(R)p(R_p|R)}{p(R_p)} = \\ = \frac{0.84 \cdot 0.77}{0.6531} = \\ = 0.990354 > 0.99 \quad (9)$$

So, the question B is answered.

The answer to the Question C.

Every car color (red, green, or blue) can be perceived as red, green, or blue, so we have a total of nine conditional probabilities. Consequently, we have a bigger table of conditional probabilities, containing nine entries, which is the following one.

	R_p	G_p	B_p
R	0.77	x	$1 - 0.77 - x$
G	0.03	y	$1 - 0.03 - y$
B	0.08	0.82	0.10

In order to answer the question we need the data of the third column of this table. The key to reveal these data is to use the first part of the information given in the current section of the story: "*the new test results accurately reconfirm the conclusion of the first test*". Let us apply the new test results for the environment of the old test. The equations (5) and (6) become

$$85 \cdot 0.77 + 15 \cdot y = 80 \quad (10)$$
$$15 \cdot 0.03 + 85 \cdot x = 20 \quad (11)$$

with the exact solutions

$$x = 0.23 \quad (12)$$
$$y = 0.97. \quad (13)$$

Consequently the previous table becomes

	R_p	G_p	B_p
R	0.77	0.23	0
G	0.03	0.97	0
B	0.08	0.82	0.10.

This implies

$$p(B_p) = \\ = p((B_p \cap R) \cup (B_p \cap G) \cup (B_p \cap B)) = \\ = p(B_p \cap R) + p(B_p \cap G) + p(B_p \cap B)) = \\ = p(R)p(B_p|R) + p(G)p(B_p|G) + p(B)p(B_p|B)) = \\ = 0.84 \cdot 0.0 + 0.13 \cdot 0.0 + 0.03 \cdot 0.08 = \\ = 0.0024 \quad (14)$$

which means that the conditional probability

$$p(B|B_p) = \frac{B \cap B_p}{p(B_p)} = \frac{p(B)p(B_p|B)}{p(B_p)} = \\ = \frac{0.03 \cdot 0.08}{0.0024} = 1.0, \quad (15)$$

hence the 100% of certainty. The last question is answered, too.

4 Conclusions

The "paradox" is fully explained by the above computations. Our paradox expressed by a problem arising in a quite elaborated environment is able to gradually sharpen the "feeling of incredible": the more the data are slack the more the conclusion drawn is reliable.

As a conclusion is interesting to see that sometimes it is not so simple to evaluate the reliability of a statement which has a quite unsure bases in an apparent way. The evaluation of such reliability could be a subtle task and in extremis sometimes there can be obtained certain information from seemingly highly uncertain assumptions.

References

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